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Engineering Notes

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First-Order Analytical Solution for Spacecraft Motion About (433) Eros

Juan F. San-Juan*
Universidad de La Rioja, 26004 Logroño, Spain
Alberto Abad†
Universidad de Zaragoza, 50009 Zaragoza, Spain
Martín Lara‡
Real Observatorio de la Armada, E-11110 San Fernando (Cádiz), Spain
and
Daniel J. Scheeres§
The University of Michigan, Ann Arbor, Michigan 48109-2140

Introduction

ORBITAL motion about asteroids is highly nonlinear because of asteroids' large shape ellipticity and rapid rotation rates. It is common to find chaotic motion in the vicinity of these celestial bodies. Previously, numerical approaches have been used to understand the dynamics of the problem. Using a numerical approach, a global criterion for the stability of motion was found in Ref. 1, where numerically determined periodic orbits were used to explore the stability of three-dimensional trajectories around asteroids. In that work it was found that families of three-dimensional periodic orbits change their stability type at certain critical inclinations, enabling the definition of a boundary—relating orbital inclination and mean radius of the orbiter—that separates regions of stable motion from unstable ones.

In contrast to an approach based on numerical solutions, the determination of action-angle variables reflecting the actual dynamics of a chaotic system enables the replacement of the (actual) nonintegrable Hamiltonian by an integrable approximation that is designed to give good agreement with the real dynamics. Such an approach has the obvious advantage of allowing the parameters of the system to be specified arbitrarily.

In this Note we obtain a first-order analytical theory of an spacecraft about a second degree and order gravity field, assumed to have a significant shape ellipticity (i.e., a relatively large C_{22} gravity coefficient). For a concrete example and for comparison with Ref. 1, we will make reference to motion about the asteroid (433) Eros, although the analysis given here is applicable to general second degree and order gravity fields, under the restrictions we will discuss later. We assume that the body is in uniform rotation around its axis of greatest inertia. This simplified model includes all of the main perturbations that act on an orbiter in this system,[2] namely, the Keplerian attraction, the Coriolis force, the oblateness, and the ellipticity perturbations.

Our analytical theory is found by averaging. First, we determine action-angle variables suitable to average the Hamiltonian over one of the fast angle variables. Then, the averaging is done by finding appropriate generating functions by means of Lie transformations. After the transformation is computed, the new Hamiltonian (in new canonical variables) appears as an explicit series depending on a small parameter. In the process of computing the canonical transformations, the influence of the angle variables is put off to higher orders of the small parameter. To this end we select the oblateness J_2 coefficient as a small parameter and use Deprit's method[3] for constructing the Lie transformations.

The usual technique when implementing closed-form analytical theories, the Delaunay normalization,[4] cannot be directly applied to our problem. To overcome this inconvenience, we first perform a simplification of the Hamiltonian making use of the Deprit's relegation algorithm[5,6] (see also Ref. 7). This procedure uses repeated iterations of a transformation after which the desired perturbation (the ellipticity perturbation in our case) appears in the new Hamiltonian with a lesser influence. When the perturbation is small enough, we can neglect it.

After the elimination of the ellipticity term, the Hamiltonian becomes equal to the Hamiltonian of the main problem of the artificial satellite, in which the longitude of the node is cyclic and, hence, the Coriolis term becomes constant and can be deleted. Then a Delaunay normalization can be performed transforming the Hamiltonian into an integrable one. The Delaunay normalization is made in closed form, without using series expansion in the eccentricity. All of these operations have been made symbolically by using the Poisson series processor (PSPC)[8] included in the software ATESAT.[9]

Finally, we note that the aim of this work is to give a qualitative description of the dynamics around Eros and not to provide an accurate ephemeris. In this sense, we find qualitative agreement with previous numerical results.[1]

Dynamical Model

To formulate the motion of a satellite around the asteroid Eros, let us consider the asteroid as a solid rotating around the z axis with constant velocity ω, and let us take up to the second order in the potential expansion. The satellite motion will be referred to a rotating frame with origin at the center of mass of Eros and whose axes coincide with its principal axes of inertia defined by the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

Under the preceding assumptions, the Hamiltonian defining the motion is

$$\mathcal{H} = \tfrac{1}{2}(\boldsymbol{X}\cdot\boldsymbol{X}) - \boldsymbol{\omega}\cdot(\boldsymbol{x}\times\boldsymbol{X}) + \mathcal{V}(\boldsymbol{x}) \tag{1}$$

where $\boldsymbol{X} = (X, Y, Z)$ are the conjugate momenta of the Cartesian variables in the rotating frame $\boldsymbol{x} = (x, y, z)$ and $\mathcal{V}$ is the potential:

$$\mathcal{V} = -\frac{\mu}{r} + \frac{\mu\alpha^2}{r^3}\left[C_{2,0}\left(\frac{1}{2} - \frac{3}{2}\frac{z^2}{r^2}\right) - 3C_{2,2}\frac{x^2 - y^2}{r^2}\right] \tag{2}$$

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*Associate Professor, Departemento de Matemáticas y Computación; juanfelix.sanjuan@dmc.unirioja.es.

†Associate Professor, Grupo de Mecánica Espacial; abad@posta.unizar.es.

‡Commander; mlara@roa.es.

§Associate Professor, Department of Aerospace Engineering; scheeres@umich.edu. Associate Fellow AIAA.

where μ is the gravitational constant, α the equatorial radius, $r = \sqrt{(x^2 + y^2 + z^2)}$ is the radial distance of the satellite, and the harmonic coefficients are $C_{2,0} < 0 < C_{2,2}$ because Eros spins around its axis of greatest inertia.

The equations of motion corresponding to the Hamiltonian (1) are

$$\dot{\boldsymbol{x}} + \boldsymbol{\omega} \times \boldsymbol{x} = \boldsymbol{X}, \qquad \dot{\boldsymbol{X}} - \boldsymbol{\omega} \times \boldsymbol{X} = -\nabla \boldsymbol{x} \mathcal{V} \tag{3}$$

and $\mathcal{H}(\boldsymbol{X}, \boldsymbol{x}) = h$ is an integral of the motion.

The simplified model of Eqs. (3) with $\mathcal{V}$ given by Eq. (2)—second degree and order gravity field and uniform rotation around the axis of greatest inertia—includes all of the main perturbations that act on an orbiter in this system,[2] namely, the Keplerian attraction, the Coriolis force, and the oblateness and ellipticity perturbations. To the well-known effects of the oblateness perturbation, the main effect of the $C_{2,2}$ term causes fluctuations in orbit energy and angular momentum that produces noticeable variations in the orbital elements.[10] When the asteroid rotates slowly as compared to the spacecraft orbit period, the averaged problem of orbital motion can be integrated in the formal sense.[11] But the situation is different for fast rotation rates, where the motion is far from being integrable because of resonances between orbital motion and rotation rates. First-order analytical theories based on averaging of the orbital elements lose accuracy, and instead of trying to give approximate solutions the analytical efforts have taken the direction of giving estimates of the stability of motion based on energy and angular momentum variations over one orbit.[2]

Ordering the Hamiltonian

Inertially referenced orbital elements have been traditionally used for studying the long-term evolution of dynamical systems. By formulating the perturbing function in orbital elements, the (averaged) Lagrange planetary equations can be integrated, providing approximate solutions for the secular motion of the satellite.

The usual averaging procedure is done by developing the perturbation function as a Fourier series in the mean anomaly ℓ with coefficients as series in powers of the eccentricity e and the inclination function $\sin I$; the validity of such solutions are constrained to small values of the eccentricity. To avoid expansions in powers of the eccentricity, we formulate the orbital problem in nodal-polar variables and carry all developments in closed form.

Defining $\boldsymbol{G} = \boldsymbol{x} \times \boldsymbol{X}$ to be the angular momentum, we materialize the ascending node by the vector $\boldsymbol{n} = \boldsymbol{k} \times \boldsymbol{G}$. The canonical nodal-polar variables (r, u, h, R, U, H), also called Whittaker or Hill variables,[12,13] are the distance r of the satellite, the argument of latitude u (the angle between $\boldsymbol{n}$ and $\boldsymbol{x}$), the argument of the node h (the angle between $\boldsymbol{i}$ and $\boldsymbol{n}$), the modulus U of $\boldsymbol{G}$, $H = \boldsymbol{G} \cdot \boldsymbol{k}$, and the radial velocity $R = \boldsymbol{X} \cdot \boldsymbol{x}/r$ in the inertial frame (see Fig. 1).

In these variables, we distinguish four terms in the Hamiltonian: the Keplerian term $\mathcal{H}_K$, the Coriolis term $\mathcal{H}_C$, the oblateness term $\mathcal{H}_o = J_2 \mathcal{H}_o^*$, and the ellipticity term $\mathcal{H}_e = J_2 \mathcal{H}_e^*$

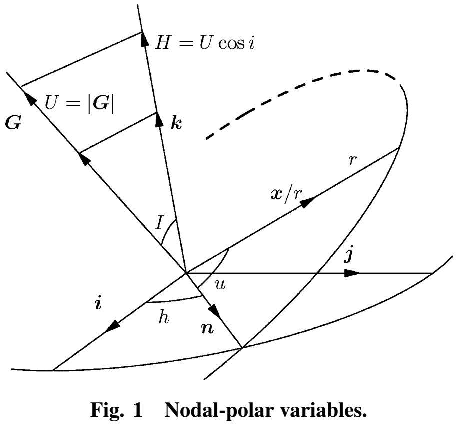

Fig. 1 Nodal-polar variables.

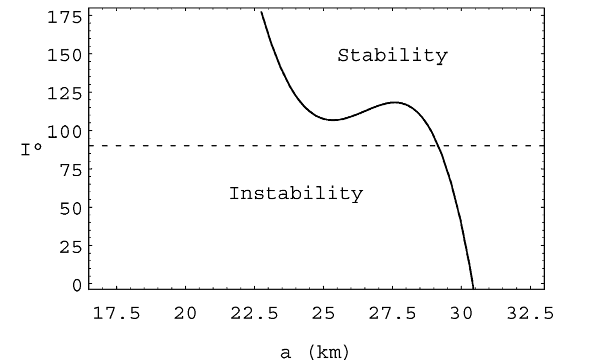

Fig. 2 Stability regions for three-dimensional motion around Eros (after Ref. 1).

$$\mathcal{H}_K = \frac{1}{2}\left(R^2 + \frac{U^2}{r^2}\right) - \frac{\mu}{r} \tag{4}$$

$$\mathcal{H}_C = -\omega H \tag{5}$$

$$\mathcal{H}_o^* = \frac{\mu \alpha^2}{4r^3}\left(2 - 3s_i^2 + 3s_i^2 \cos 2u\right) \tag{6}$$

$$\mathcal{H}_e^* = \frac{\mu \alpha^2 C_{2,2}'}{4r^3}\left[12 s_i^2 \cos 2h - 3\left(2 - 2c_i - s_i^2\right)\cos(2u - 2h) - 3\left(2 + 2c_i - s_i^2\right)\cos(2u + 2h)\right] \tag{7}$$

where $c_i = \cos I$ and $s_i = \sin I$ are functions of the momenta U, H, and we take the usual convention $J_2 = -C_{2,0}$, and $C_{2,2}$ is replaced by $C_{2,2}' = C_{2,2}/J_2$.

To check the relative influence of these four terms in the Hamiltonian, we resort to the global criterium for stability given in Ref. 1, where the stability characteristics of low-eccentricity orbits are shown to depend on their inclination. Thus, in Fig. 2 the line

$$I \text{ deg} = 1447.87a - 163.846a^2 + 6.19364a^3 - 0.0779807a^4 \tag{8}$$

where the semimajor axis a must be in kilometers, separates stable, almost circular motion from unstable ones.

We compute the respective values of Eqs. (4–7) for a wide range of initial conditions sweeping part of the stability area provided in Fig. 2; more precisely, we examine the nonchaotic region corresponding to (stable) almost circular periodic motion further than 26 km away from the center of mass of Eros. We conclude that the sum of the Keplerian term plus the Coriolis term dominates the dynamics, while the influence of the oblateness and the ellipticity perturbations, both of the same order, is lower and remains at a higher order. Therefore, at least in that region, the Hamiltonian admits the following asymptotic expansion:

$$\mathcal{H} = \mathcal{H}_o + \epsilon \mathcal{H}_1 = (\mathcal{H}_K + \mathcal{H}_C) + \epsilon\left(\mathcal{H}_o^* + \mathcal{H}_e^*\right) \tag{9}$$

in which we select $\epsilon = J_2$ as the small parameter. It is significant to note that the influence of the ellipticity term remains at first order, which is different from the Earth artificial satellite problem.

Now we proceed in computing the Lie transformations that will provide our analytical theory. Details about the method can be found in Ref. 3.

Relegation of the Longitude of the Node

The relegation of the node tries to eliminate the angle h by using a transformation after which the desired perturbation appears in the new Hamiltonian with a lesser influence. After repeated iterations of the transformation, the perturbation should be small enough so that we can neglect it.

In our case, the longitude of the node appears only in the ellipticity perturbation, Eq. (7). Thus, one can look for Lie transformations that increase the power m in the factor $(1/r)^m$. After three iterations, we find in the new Hamiltonian that

$$\mathcal{H}_e^{*\prime} = \frac{3\mu\alpha^2 C'_{2,2}}{16\omega^3 r^6} \times \left[U(c_i - 1)^2 \left(\frac{15U^2}{r^3} - \frac{11\mu}{r^2} - \frac{60R^2}{r} \right) \cos(2u - 2h) + U(c_i + 1)^2 \left(\frac{15U^2}{r^3} + \frac{11\mu}{r^2} + \frac{60R^2}{r} \right) \cos(2u + 2h) - R\left(c_i^2 - 1\right) \left(\frac{45U^2}{r^2} - \frac{45\mu}{r} - 60R^2 \right) \sin 2h - R(c_i - 1)^2 \left(\frac{105U^2}{2r^2} - \frac{21\mu}{r} - 30R^2 \right) \sin(2u - 2h) + R(c_i + 1)^2 \left(\frac{105U^2}{2r^2} - \frac{21\mu}{r} - 30R^2 \right) \sin(2u + 2h) \right] \tag{10}$$

At zeroth order, $R = (Ue/p)\sin f$, $\mu = U^2/p$, and $U = r^2\dot{u}$, where f is the true anomaly and $p = a(1 - e^2)$. Then, Eq. (10) has a coefficient that is approximately $(\dot{u}/\omega)^3$ (see Ref. 7 for details). For $a > 26$ km, where Eq. (9) applies, this coefficient is of the same order of the Eros' J_2. Thus, for the accepted values of Eros,[14] for $a = 26$ km one finds that $(\dot{u}/\omega)^3 \approx 0.11 \approx J_2$. Therefore, the contribution of $\mathcal{H}_e^{*\prime}$ will remain at second order of J_2, and it can be neglected from our first-order approach.

Note also that the term $\mathcal{H}_o^*$ is not affected by the transformation because it does not depend on the argument of the node.

Thus, at first order of ϵ, after the relegation of the node we obtain the Hamiltonian

$$\mathcal{K} = \mathcal{K}_0 + \epsilon\mathcal{K}_1 = \mathcal{H}_K + \mathcal{H}_C + \epsilon\mathcal{H}_o^* \tag{11}$$

where the $\mathcal{H}$ are the ones given in Eqs. (4–6), but now expressed in new variables. The argument of the node h becomes a cyclic variable, and, therefore, $\mathcal{H}_C$ represents a constant of the motion that can be dropped from the new Hamiltonian $\mathcal{K}$. Then, the transformed Hamiltonian is $\mathcal{H}_K + \epsilon\mathcal{H}_o^*$, which is formally equivalent to the Hamiltonian of the main problem of an Earth satellite but, of course, with different values for the constants. Therefore, taking into account the scale invariance of the problem[15] the dynamics should be the well-known dynamics of the main problem of the artificial satellite, except for a different length scale corresponding to the quantitative value of the Eros's J_2.

We must note that the main problem of the artificial satellite enjoys cylindrical symmetry, which is not the case for an asteroid in general. Consequently, the validity of our theory will be limited to certain regions of phase space, more precisely for the case of Eros, for orbits further than ≈ 30.2 km. This is the limiting distance for which stable almost circular motion exists for all inclinations[1] and can be seen in Fig. 2 as the origin ($I = 0$ deg) of the line that separates stable and unstable almost circular motion. More general analytical theories that must include the ellipticity effect will be the topic of a future paper.

Delaunay Normalization

For convenience, we use the Delaunay variables: the mean anomaly ℓ, the argument of the pericenter g, the argument of the node h, and the conjugated momenta

$$L = \sqrt{\mu a}, \qquad G = L\sqrt{1 - e^2}, \qquad H = G\cos I \tag{12}$$

Note that $U = G$, and $u = g + f$.

Then, we write the asymptotic Hamiltonian (11)

$$\mathcal{K} = -\frac{\mu^2}{2L^2} + \epsilon\frac{\mu\alpha^2}{4G^2 r^3}[3H^2 - G^2 + 3(G^2 - H^2)\cos 2(g + f)] \tag{13}$$

where r, f are implicit functions of ℓ.

The Delaunay normalization[4] is a Lie transformation that maps the Hamiltonian

$$\mathcal{K} = \sum_{n\geq 0} \left(\frac{\varepsilon^n}{n!}\right)\mathcal{K}_n$$

into a new one

$$\mathcal{R} = \sum_{n\geq 0} \left(\frac{\varepsilon^n}{n!}\right)\mathcal{R}_n$$

in which $\mathcal{R}_n$ belongs to the kernel of the Lie derivative, that is, $\mathcal{R}_n$ is the average of $\mathcal{K}_n$ over the mean anomaly ℓ.

Instead of using the expansions of r and f in powers of the eccentricity, we will compute the integrals with respect to ℓ. But, taking into account the relation

$$a\sqrt{1 - e^2}\,\mathrm{d}\ell = r\,\mathrm{d}f$$

we change the independent variable to be the true anomaly f. Then, at first order we find that ℓ and g simultaneously disappear, and the Hamiltonian after Delaunay normalization is

$$\mathcal{R} = -\frac{\mu^2}{2L^2} + \epsilon\alpha^2\mu^4\frac{3H^2 - G^2}{4G^5 L^3}$$

which depends only on the momenta and which is trivially integrable.

Conclusions

For highly nonlinear dynamical systems, analytical theories can be constructed using suitable action-angle variables. In the case of motion about an asteroid, after relegating the node we obtain the Hamiltonian of the main problem of the artificial satellite—a well-known problem. Then, we perform a Delaunay normalization and arrive at an integrable Hamiltonian that solves the problem. This result shows the formal equivalence of motion about an arbitrary mass distribution and motion about an oblate body when far enough from the centerbody. It does not address the problem when resonant interactions with the rotating body are present.

The relegation algorithm is not a new technique and has been applied to the Earth artificial satellite problem where the J_2 perturbation is at first-order and the tesseral harmonics are treated as second-order effects. In contrast, in this work the contribution of Eros' $C_{2,2}$ and J_2 effects must be considered at the same order.

The first-order solution of this paper shows qualitative agreement with previous numerical results, and therefore it can be considered as a first step in the procedure of computing higher-order analytical theories more suitable for computing accurate ephemeris.

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Jacobi Pseudospectral Method for Solving Optimal Control Problems

Paul Williams*
RMIT University, Melbourne, Victoria, 3001 Australia

Introduction

THE solution of optimal control problems via direct methods has become popular in recent years and has been used to solve a wide variety of problems successfully.[1] The most common direct methods can be grouped into 1) local and 2) global methods. Local methods have been termed direct collocation[2] or direct transcription.[3] In local methods, a series of node points of arbitrary spacing are defined at which both the state and control vectors are collocated. The state equations are enforced as equality constraints at internal collocation points between the nodes by implicit integration techniques such as Simpson's rule (see Ref. 4) or by Gauss–Lobatto quadrature rules (see Ref. 2). In other words, the state equations are enforced locally. Global or pseudospectral methods use globally orthogonal interpolating polynomials based on the Gauss–Lobatto points for Legendre (see Refs. 5 and 6) or Chebyshev (see Refs. 7 and 8) polynomials to approximate the state and control variables. The state equations are enforced by differentiating the approximating polynomial at the corresponding Gauss–Lobatto points (which are the zeros of the derivative of the interpolating polynomial), rather than through numerical integration. Global methods are expected to be more accurate than local methods because the discrete adjoint multipliers retain the same order of accuracy as the state equations, which is not true for certain classes of local methods [such as the Hermite–Simpson (see Ref. 3) and particular Runge–Kutta discretizations (see Ref. 9)]. Ross and Fahroo[10] discuss these ideas in detail.

The use of pseudospectral methods for solving optimal control problems has been restricted in the literature to either Legendre or Chebyshev methods. In this Note, the pseudospectral method is generalized to consider collocation based on the roots of the derivatives of general Jacobi polynomials. The Legendre and Chebyshev nodes can be obtained as particular cases of the more general formulation. Comparisons between different nodes can be obtained by simply changing the parameters in the Jacobi polynomial. Appropriate selection of the Jacobi parameters may form an important part in determining real-time solutions to nonlinear optimal control problems.

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*Ph.D. Candidate, Department of Aerospace Engineering. Student Member AIAA.

Numerical Method

Statement of the Problem

Consider the problem of minimizing the performance index

$$\mathcal{J} = \phi[\ddot{\boldsymbol{x}}(t_f), \dot{\boldsymbol{x}}(t_f), \boldsymbol{x}(t_f), t_f] + \int_{t_0}^{t_f} [\mathcal{L}(\ddot{\boldsymbol{x}}(t), \dot{\boldsymbol{x}}(t), \boldsymbol{x}(t), \boldsymbol{u}(t), t)]\,\mathrm{d}t \tag{1}$$

where $t \in R$, $\boldsymbol{x} \in R^n$, and $\boldsymbol{u} \in R^m$ are subject to the dynamic constraints

$$f[\ddot{\boldsymbol{x}}(t), \dot{\boldsymbol{x}}(t), \boldsymbol{x}(t), \boldsymbol{u}(t), t] = \boldsymbol{0}, \qquad t \in [t_0, t_f] \tag{2}$$

and boundary conditions

$$\boldsymbol{\psi}_0[\dot{\boldsymbol{x}}(t_0), \boldsymbol{x}(t_0), t_0] = \boldsymbol{0} \tag{3}$$

$$\boldsymbol{\psi}_f[\dot{\boldsymbol{x}}(t_f), \boldsymbol{x}(t_f), t_f] = \boldsymbol{0} \tag{4}$$

where $\boldsymbol{\psi}_0 \in R^p$ and $\boldsymbol{\psi}_f \in R^q$, with $p \le n$ and $q \le n$, and the state and control constraints

$$\boldsymbol{g}[\dot{\boldsymbol{x}}(t), \boldsymbol{x}(t), \boldsymbol{u}(t)] \le \boldsymbol{0}, \qquad \boldsymbol{g} \in R^r \tag{5}$$

Discretization of the State Equations

The Jacobi pseudospectral method for solving optimal control problems, like the Legendre and Chebyshev methods, is based on expanding the state and control trajectories using Lagrange interpolating polynomials. An arbitrary selection of node points can lead to poor interpolation characteristics such as the Runge phenomenon, and so the node points in pseudospectral methods are chosen as the Gauss–Lobatto points. In this Note, the nodal points are obtained as the extrema of the Nth order Jacobi polynomial $P_N^{(\alpha,\beta)}$, where $\alpha > -1$ and $\beta > -1$ are parameters that determine the characteristic of the polynomial. Jacobi polynomials are orthogonal over the interval $(-1, 1)$ with respect to the weight function $w(\tau) = (1-\tau)^\alpha (1+\tau)^\beta$ and are the eigenfunctions of the Sturm–Liouville problem (see Ref. 11)

$$(1-\tau^2) P_N''^{(\alpha,\beta)} + [(\beta - \alpha) - (\alpha + \beta + 2)\tau] P_N'^{(\alpha,\beta)} + N(N + \alpha + \beta + 1) P_N^{(\alpha,\beta)} = 0 \tag{6}$$

The Legendre and Chebyshev polynomials belong to the class of Jacobi polynomials and may be obtained by setting $\alpha = \beta = 0$ for Legendre and $\alpha = \beta = -0.5$ for Chebyshev.